Noncommutative geometry of the Satake compactification

Correspondences in Istanbul

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Harish-Chandra and the Plancherel formula

Plancherel's formula, from early in the last century, asserts that

$$\int_{\mathbb{R}} |f(x)|^2\, dx = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \frac{d\xi}{2\pi} \qquad \left(\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{ix\xi}\, dx \right)$$

An early accomplishment of **C*-algebra theory** was the explanation of when a generalization to other groups G, beyond $\mathbb{R}$, is possible …

$$\|f\|^2_{L^2(G)} = \int_{\widehat{G}} \|\pi(f)\|^2_{H.S.}\, d\mu_G(\pi) \qquad \left(\pi(f) = \int_G f(g)\pi(g)\, dg \right)$$

Harish-Chandra made this abstract formula explicit for real reductive groups, such as $G = SL(2,\mathbb{R})$:

$$\|f\|^2_{L^2(G)} = \sum_{n\neq 0} \|\pi_n(f)\|^2_{H.S.} \cdot |n| + \int_0^\infty \|\pi_{\mathrm{ev},s}(f)\|^2_{H.S.} \cdot \frac{s}{2} \tanh\left(\frac{\pi s}{2}\right) ds$$

$$+ \int_0^\infty \|\pi_{\mathrm{odd},s}(f)\|^2_{H.S.} \cdot \frac{s}{2} \coth\left(\frac{\pi s}{2}\right) ds$$

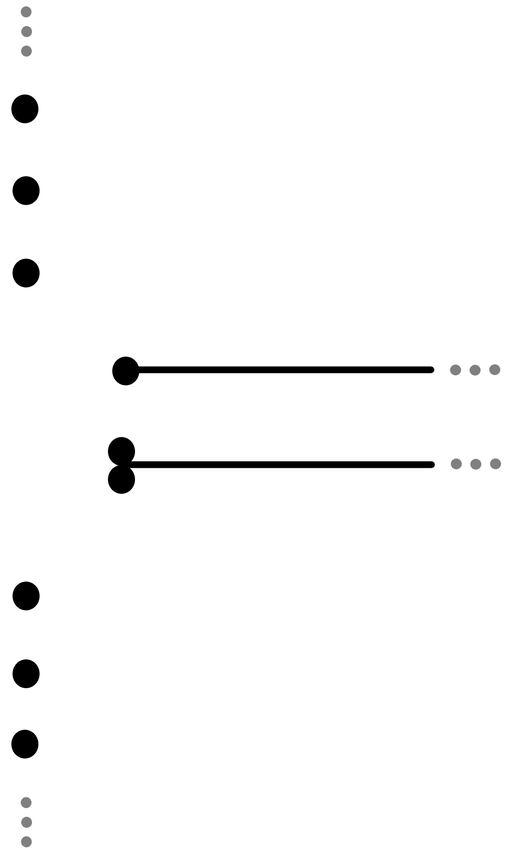
The tempered dual of a real reductive group

Harish-Chandra's **real reductive Lie groups** are (more or less) the closed, connected subgroups of $GL(N, \mathbb{R})$ that are invariant under the transpose operation.

His **tempered irreducible unitary representations** are (more or less) those that are used in his Plancherel formula.

Harish-Chandra's precise definition determines a closed subset of the unitary dual, the **support of the Plancherel measure**.

The tempered unitary dual is also the **spectrum of the reduced group C*-algebra** $C_r^*(G) \subseteq \mathfrak{B}(L^2(G))$



Two principles underlying the Plancherel formula

Harish-Chandra discovered quite early on the principles which allowed him to do this [obtain an explicit Plancherel formula] but he overcame the obstacles to their proof only very slowly ...

The first principle is that the **representations** [**parabolically**] **induced from ... square-integrable** [**representations**] **suffice** for the Plancherel formula ...

The second is that [a group G] has square-integrable representations if and only if there are [compact] Cartan subgroups ...

Robert Langlands
Review of Harish-Chandra's Collected Works

The purpose of this lecture, and Jacob's lecture, is to study the first principle.

The second principle has been extensively studied from the geometric point of view (via index theory, by Parthasarthy, Atiyah-Schmid, Lafforgue).

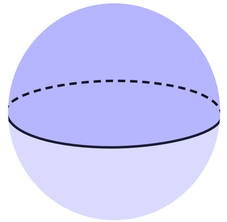
Introducing the Satake compactification

Let G be a real reductive group with **compact center**. Let K be a maximal compact subgroup of G.

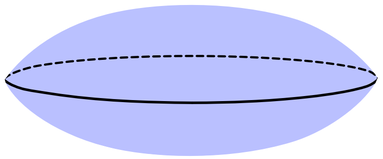
The **Satake compactification** (Ichiro Satake, 1960) is a G-equivariant compactification $\mathfrak{X}$ of the symmetric space G/K.

There are many such compactifications. This particular one …

- … has finitely many G orbits
- … is a smooth G-manifold with corners



$SL(2,\mathbb{R})$



$SL(3,\mathbb{R})$

Our general aim is to relate the strata of the Satake compactification to parts of the tempered dual of G.

A definition of the Satake compactification

Lemma (Fell, Hausdorff,…). *The set of all non-empty, closed subsets of a locally compact space is itself a* ***locally compact Hausdorff space****.*

Lemma (Fell, Chabauty). *The set of all closed subgroups of a locally compact group is a* ***compact Hausdorff space***.

Definition. Let G be a real reductive group. The associated (maximal) **Satake compactification** is the closure, in the space of all closed subgroups, of the set of all maximal compact subgroups.

(Compactification of what? All maximal compact subgroups are conjugate to one another, so the set of all maximal compact subgroups is a homogeneous space of G, and a symmetric space.)

A brief look at integral operators

> In the winter of 1900-1901 the Swedish mathematician E. Holmgren reported in Hilbert's seminar on Fredholm's first publications on integral equations, and it seems that **Hilbert caught fire at once**...
>
> Hermann Weyl
> David Hilbert and his mathematical work

As everyone knows, Hilbert transformed the theory of integral equations, especially the theory of the eigenvalue equation

$$\int k(x, y) f(y)\, dy = \lambda f(x),$$

with $k(x, y)$ defined, and e.g. smooth or continuous, on a compact space.

For the most part, his focus was on the spectral theory of **individual operators** …

Algebras of integral operators

> The story would be dramatic enough had it ended there, but then a sort of miracle happened ...
>
> Hermann Weyl
> David Hilbert and his mathematical work

Weyl is referring to quantum mechanics, but the words apply equally well to developments spearheaded by Alain Connes, starting in the late 1970's, first in the context of foliations.

The idea: study **algebras** of integral operators associated to smooth **integral kernels defined on pairs** (x, y) **for which** x **and** y **belong to the same leaf** using

$$(k_1 \star k_2)(x, y) = \int_L k_1(x, z) k_2(z, y)\, dz \qquad (x, y \in L)$$

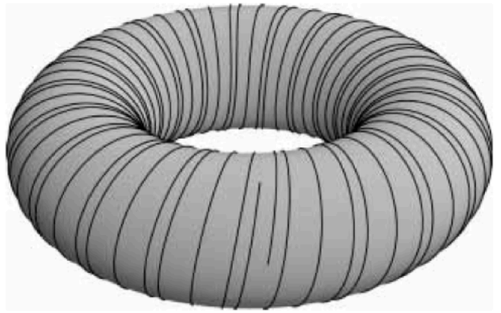


Image credit: Walter van Suijlekom

Groupoids and algebras of integral operators

The formula

$$(k_1 \star k_2)(x, y) = \int_L k_1(x, z) k_2(z, y)\, dz$$

is easily adapted to other contexts. Easiest and most relevant here is the case of a **Lie groupoid**, where the formula reads

$$(k_1 \star k_2)(\alpha) = \int_{G^{\mathrm{source}(\alpha)}} k_1(\alpha \circ \beta) k_2(\beta^{-1})\, d\beta$$

This formula defines an associative product on smooth compactly supported functions on a Lie groupoid. The Lie groupoid context covers

- composition of classical integral operators
- the foliation case
- the convolution product in the group case

Omar Mohsen's groupoid

Let G be a group. Any group. And let $\{S\}$ be a **collection of subgroups** that is **closed under conjugation** by elements of G. The set $\{C\}$ of **all cosets** of all the subgroups in $\{S\}$ is a groupoid, with

$$\text{source}(C) = C^{-1} \cdot C \qquad \text{target}(C) = C \cdot C^{-1} \qquad C_1 \circ C_2 = C_1 \cdot C_2$$

Now, when G is a real reductive group, and $\{S\} = \mathfrak{X}$ is the Satake compactification, we may give the set of cosets $\{C\} = G_{\mathfrak{X}}$, which we'll call the **Satake groupoid**, the topology it inherits as a collection of nonempty closed subsets of G.

Lemma. *$G_{\mathfrak{X}}$ is a locally compact Hausdorff topological groupoid.*

Three faces of the Satake groupoid

The Satake compactification $\mathfrak{X}$ and its groupoid $G_{\mathfrak{X}}$ can be defined very quickly via the Fell topology. But it is difficult to study $G_{\mathfrak{X}}$ this way.

The Corleck Head. Wikipedia

But there are other quite different approaches to $G_{\mathfrak{X}}$. A second approach is due to **Toshio Ōshima**, who gave a remarkable Lie-theoretic account of $\mathfrak{X}$.

A third approach is purely geometric, more or less. Since $\mathfrak{X}$ is a **manifold with corners**, there is a pre-existing construction of an associated groupoid (*Monthubert et al, following Melrose*). I won't discuss this.

Ōshima's approach to the Satake compactification

Ōshima's construction uses an **Iwasawa decomposition** $G = KAN$ (and a little bit more).

I'll describe the construction for $G = SL(2,\mathbb{R})$, where

$$K = SO(2), \quad A = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} : \alpha > 0 \right\}, \quad N = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{R} \right\}$$

Let $\mathbf{G} = G \times \mathbb{R}$. Form the quotient $\mathbf{G}/\mathbf{H}$ by the smooth family $\mathbf{H}$ of subgroups

$$H_t = \left\{ \begin{bmatrix} c & st^2 \\ -s & c \end{bmatrix} : c^2 + s^2 t^2 = 1 \right\} \qquad (t \in \mathbb{R})$$

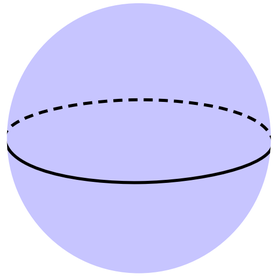
Note that $\mathrm{Ad}_a[H_t] = H_{a \cdot t}$, where $a \cdot t = \alpha^2 t$. The **Oshima manifold**, a smooth closed G-manifold, is $\mathfrak{M} = (\mathbf{G}/\mathbf{H})/A$ (quotient by the diagonal right action of A).

Ōshima's construction and the Satake compactification

The Oshima manifold, again: $\mathfrak{M} = (\mathbf{G}/\mathbf{H})/A$.

We have been discussing $G = SL(2,\mathbb{R})$, and in that case the map

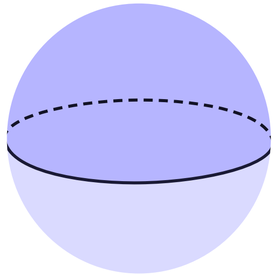
$$\mathfrak{M} \longrightarrow \mathbb{C}_\infty$$
$$[\![g,t]\!] \longmapsto g \cdot ti$$



(action by Möbius transformations on the Riemann sphere) is a diffeomorphism of smooth, closed G-manifolds.

The subgroup $H_{[\![g,t]\!]} = \mathrm{Ad}_g[H_t]$ of G depends smoothly on $[\![g,t]\!] \in \mathfrak{M}$. The map

$$\{\, [\![g,t]\!] \in \mathfrak{M} : t \geq 0 \,\} \longrightarrow \mathfrak{X}$$
$$[\![g,t]\!] \longmapsto H_{[\![g,t]\!]}$$



is a homeomorphism.

The Ōshima groupoid and the Satake groupoid

The smooth family $H_{\mathfrak{M}} = \{ H_{[\![g,t]\!]} : [\![g,t]\!] \in \mathfrak{M} \}$ of subgroups of G can be viewed as a normal subgroupoid of the transformation groupoid $G \ltimes \mathfrak{M}$

Definition. The **Ōshima groupoid** is the quotient groupoid

$$G_{\mathfrak{M}} = G \ltimes \mathfrak{M} \Big/ H_{\mathfrak{M}}$$

This is a Lie groupoid whose object space is the Ōshima space $\mathfrak{M}$.

Restricted to the Satake compactificaton $\mathfrak{X} \subseteq \mathfrak{M}$, the Oshima groupoid identifies with the Satake groupoid as follows:

$$\big(g_1, [\![g_2, t]\!]\big) \longmapsto g_1 H_{[\![g_2,t]\!]}$$

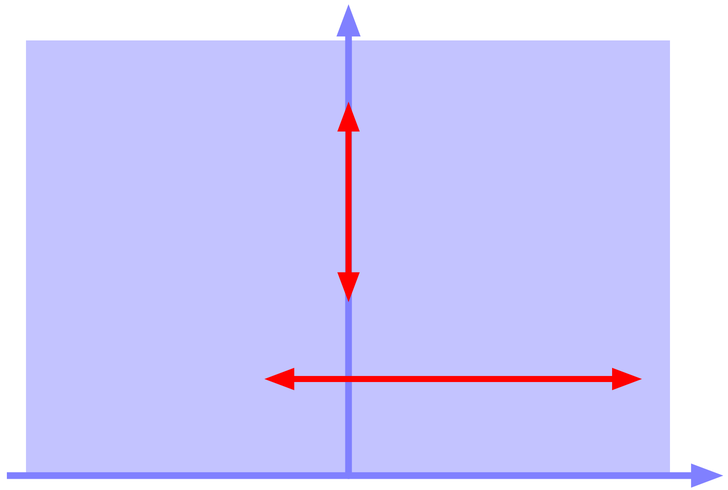
Local structure (the big cell)

Reductive groups are complicated. But **the Satake groupoid is very simple**!

For example, for $G = SL(2,\mathbb{R})$, on the **big cell**, away from $\infty \in \mathbb{C}_\infty$, the Satake groupoid is

$$\big(N \times N\big) \times \big(A \ltimes [0,\infty)\big)$$

Pair groupoid *Action groupoid*



$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{bmatrix} \cdot i = x + iy$$

The Satake groupoid offers a **hugely simplified view** of G and its action on $\mathfrak{X}$ (e.g. away from the boundary, it is just the pair groupoid). This is the point of the Satake groupoid!

Parabolic induction

Back to representation theory …

Parabolic induction is a **transfer operation** that constructs unitary reps of a group like $G = SL(n, \mathbb{R})$ from unitary reps of a (proper) **block diagonal subgroup**.

Let L be a block-diagonal subgroup and let $P = LN$ be the corresponding **standard parabolic subgroup** of block upper triangular matrices.

Transfer is defined using a **C*-correspondence** (*work of Pierre Clare*): the test functions on G/N generate a Hilbert $C_r^*(L)$-module $C_r^*(G/N)$ that carries a left-action of $C_r^*(G)$ with

$$C_r^*(G) \longrightarrow \mathfrak{K}\Big(C_r^*(G/N)\Big).$$

An important feature: transfer **almost always takes irreps to irreps** (*work of Bruhat*). And the transfer of an irrep **always** decomposes into a finite sum of irreps.

The cuspidal ideal

The following terminology is borrowed from Harish-Chandra:

Definition. The **cuspidal ideal** of $C_r^*(G)$ is

$$C^*_{\text{cuspidal}}(G) = \bigcap_N \ker\left(C_r^*(G) \to \mathfrak{K}(C_r^*(G/N)\right)$$

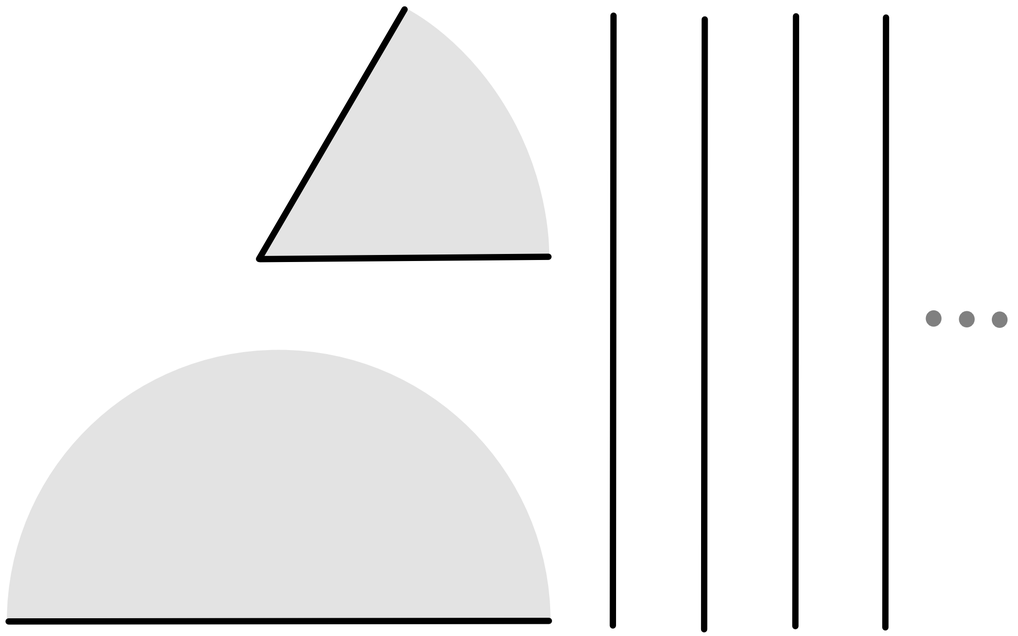
Lemma. *An irreducible tempered representation of G embeds in a parabolically induced representation if and only if it vanishes on the cuspidal ideal.*

The question is, **what are the other tempered irreducible representations** of G?
By C*-algebra theory, these are the irreps of $C^*_{\text{cuspidal}}(G)$ **...**

The role of the Satake groupoid and its C*-algebra: demystify $C^*_{\text{cuspidal}}(G)$.

More on the shape of the tempered dual

Before leaving the topic of parabolic induction, let's look again at the form of the tempered dual.



Here is the tempered dual of $G = SL(3,\mathbb{R})$.

There are two 2-dimensional minimal principal series, together with an infinite family of intermediate principal series.

The shapes of these families are *not* determined by the two Harish-Chandra principles mentioned by Langlands

Instead they are determined by certain finite intertwining groups associated to G. The role these groups play amounts to a **third** Harish-Chandra principle …

Bernstein's basic geometric lemma

Joseph Bernstein gave a geometric account of the third principle for **smooth representations of reductive p-adic groups**. Here is part of the story, as it applies the the **minimal parabolic subgroup** $P = LN$ (the other parts require more notation).

Write $\mathcal{S}(\ \ ) =$ locally constant, compactly supported functions and form

$$\mathscr{E} = \mathcal{S}(N\backslash G) \otimes_{\mathcal{S}(G)} \mathcal{S}(G/N)$$

Now, there is a **Frobenius reciprocity** isomorphism for the functor Ind of minimal parabolic induction:

$$\mathrm{Hom}_{\mathcal{S}(G)}\big(\mathrm{Ind}\, V, \mathrm{Ind}\, W\big) \cong \mathrm{Hom}_{\mathcal{S}(L)}\big(\mathscr{E} \otimes_{\mathcal{S}(L)} V, W\big),$$

and by the **Bruhat decomposition** $G = \sqcup_w PwP$, there is a decomposition of $\mathscr{E}$ into composition factors $\mathcal{S}(L)_{\mathrm{Ad}_w}$.

The basic geometric lemma for tempered representations of real groups

Can we do for tempered representations of real groups what Bernstein did for smooth representations of p-adic groups (and ultimately build a "tempered Bernstein center")?

Maybe …

- The representation-theoretic consequences are all true (*work of Langlands*).
- There *is* a Frobenius reciprocity isomorphism (*Tyrone Crisp & NH*). But $C_r^*(G/N)$ is not (transparently) a Hilbert $C_r^*(G)$-module, so operator space methods seem to be required.
- But perhaps one should approach the Bruhat theory in a more wholeheartedly noncommutative-geometric way …

Compact representations and the compact ideal

Back to the Harish-Chandra's first principle …

Recall that we had defined the ideal $C^*_{\text{cuspidal}}(G) \lhd C^*_r(G)$. Here is another:

Definition. The **compact ideal** in $C^*_r(G)$ is

$$C^*_{\text{compact}}(G) = \left\{ f \in C^*_r(G) : \lambda(f) \in \mathfrak{K}\left(L^2(G/K; V)\right) \ \forall V \in \widehat{K} \right\}$$

Definition. An irreducible tempered representation of G is *compact* if it is **admissible** and if $\{\pi\}$ is an **open and closed subset of the tempered dual**.

Theorem. *The spectrum of the compact ideal is the set of compact representations of G (and in particular consists only of discrete series).*

Harish-Chandra's principle and ideals in the reduced C*-algebra

Thanks to the above, Harish-Chandra's principle can be expressed purely ideal-theoretically (actually the following is stronger than Harish-Chandra's principle):

Theorem. *If G is a real reductive group with compact center, then*

$$C^*_{\mathrm{cusp}}(G) = C^*_{\mathrm{compact}}(G)\,.$$

Jacob will discuss the proof in his lecture. But in a nutshell, the proof is only a matter of studying the extension of groupoid C*-algebras

$$0 \to C^*_r(G_{\mathrm{int}\,\mathfrak{X}}) \to C^*_r(G_{\mathfrak{X}}) \to C^*_r(G_{\partial\mathfrak{X}}) \to 0.$$

- $C^*_r(G_{\mathrm{int}\,\mathfrak{X}}) \cong \mathfrak{K}(L^2(G/K))$
- $C^*_r(G_{\partial\mathfrak{X}})$ is assembled from the C*-algebras $\mathfrak{K}(C^*_r(G/N))$ in a very simple way

Summary and outlook

The Satake groupoid, invented by Omar Mohsen, and its C*-algebra, despite being huge simplifications of G and its reduced C*-algebra, still retain enough of the structure of G and $C_r^*(G)$ to be very useful in representation theory.

Naturally we are looking for other uses (if you have a hammer …)

- The issue of intertwining groups?
- The Langlands classification?
- Spherical Plancherel formulas à la Bernstein, Sakellaridis-Venkatesh?
- More?

Thank you very much!

Congratulations, and many happy returns, Georges!